

Girard couples of quantales

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Abstract

We introduce the concept of a *Girard couple*, which consists of two (not necessarily unital) quantales linked by a strong form of duality. The two basic examples of Girard couples arise in the study of endomorphism quantales and of the spectra of operator algebras. We construct, for an arbitrary sup-lattice S , a Girard quantale whose right-sided part is isomorphic to S .

1 Introduction

Girard quantales were introduced by Yetter to provide semantics for a certain fragment of non-commutative linear logic known as *cyclic linear logic* [5, 17]. They are, essentially, quantales with a well-behaved negation operation, and therefore play a role among all quantales analogous to that played by complete boolean algebras among frames. They are also related to the much older notion of *MV-algebra* [2, 11].

Endomorphisms quantales $\mathcal{Q}(S)$ have been studied by C. J. Mulvey and J. Wick Pelletier as quantales of *linear relations* [9]. These quantales admit a “von Neumann duality” between their right- and left-sided elements; but this can not, in general, be extended to arbitrary elements of the quantale.

In this paper, we construct a *predual quantale* $\mathcal{C}(S)$ for the endomorphism quantale $\mathcal{Q}(S)$ and show that the pairs $(\mathcal{C}(S), \mathcal{Q}(S))$ enjoy properties among all suitable pairs of quantales which are analogous to those of a Girard quantale. In particular, we define a negation operation which extends the existing von Neumann duality.

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Our construction is reminiscent of one arising in functional analysis, where the ideal of trace-class operators on a Hilbert space is the (Banach space) predual of the algebra of all bounded operators on that Hilbert space. By considering appropriate topologies on the preceding algebras, we can construct further fundamental examples of what we shall call *Girard couples*.

We also note that there is a characterisation of Girard couples in terms of monoidal functors which, in turn, suggests further generalisations: by considering more complex “gradings” of quantale structures, or by replacing sup-lattices by objects of an other $*$ -autonomous category.

2 Preliminaries

We review some of the basic definitions and results of quantale theory which will be extensively used in the sequel. Details may be found in [1, 6, 7, 8, 10, 16].

The category of complete lattices and supremum-preserving maps will be denoted \mathbf{Sup} ; we shall follow the convention of referring to objects and arrows of \mathbf{Sup} as *sup-lattices* and *sup-homomorphisms*, respectively. The top and bottom elements of a sup-lattice will be denoted $1, 0$, respectively. We say that a sup-homomorphism is *strong* if it preserves the top element.

The category \mathbf{Sup} has a $*$ -autonomous structure: the *tensor product* of sup-lattices S and T , denoted $S \otimes T$, is the free sup-lattice with generators $\{s \otimes t \mid s \in S, t \in T\}$ satisfying the relations

$$\bigvee (s_i \otimes t) = \left(\bigvee s_i \right) \otimes t \qquad \bigvee (s \otimes t_j) = s \otimes \left(\bigvee t_j \right)$$

for all $s, s_i \in S, t, t_j \in T$; the *tensor unit* is the two-element chain $\mathbf{2} = \{0, 1\}$; the *dual* of a sup-lattice S is simply its opposite, denoted S^{op} .

We mark elements of S^{op} with $'$ whenever the distinction from elements of S is desirable. Every sup-homomorphism $f : S \rightarrow T$ has a right adjoint $f^{-1} : T \rightarrow S$ which preserves arbitrary infima, and so may be regarded as a sup-homomorphism $f^* : T^{\text{op}} \rightarrow S^{\text{op}}$, $f^*(x') = f^{-1}(x)'$; this is the *dual* of f .

A *quantale* is a sup-lattice Q equipped with an associative multiplication that distributes joins. The right adjoints of $a \cdot ()$ and $() \cdot a$ are denoted $() \leftarrow a, a \rightarrow ()$, respectively; they can be computed as below.

$$b \leftarrow a = \bigvee \{c \mid ac \leq b\} \qquad a \rightarrow b = \bigvee \{c \mid ca \leq b\}$$

An element $r \in Q$ is said to be *right-sided* if $r1 \leq r$. Similarly, $l \in Q$ is *left-sided* if $1l \leq l$. A *two-sided* element is both right- and left-sided. The

sets of right-, left-, and two-sided elements are denoted $\mathcal{R}(Q)$, $\mathcal{L}(Q)$ and $\mathcal{T}(Q)$, respectively. Note that the *left annihilator* $a \rightarrow 0$ of any $a \in Q$ is left-sided and that the *right annihilator* $0 \leftarrow a$ is right-sided. Thus the mappings $(\) \rightarrow 0, 0 \leftarrow (\)$ establish a pseudoduality between $\mathcal{R}(Q)$ and $\mathcal{L}(Q)$. We write $l \perp r \Leftrightarrow lr = 0, r^\perp = r \rightarrow 0, l^\perp = 0 \leftarrow l$.

We say that Q is: *unital* if it has a neutral element, *i.e.*, an $e \in Q$, such that $ae = ea = a$ for all $a \in Q$; *semiunital* if $r1 = r, 1l = l$ for all $r \in \mathcal{R}(Q), l \in \mathcal{L}(Q)$; *von Neumann* if $^\perp$ is a duality between $\mathcal{R}(Q)$ and $\mathcal{L}(Q)$. In a semiunital quantale Q , $1a \geq a$ and $a1 \geq a$ hold for every $a \in Q$.

A *homomorphism of quantales* is a sup-homomorphism preserving the multiplication. It is *unital* if it also preserves the neutral element.

A *left Q -module* is a sup-lattice M together with an action of Q on M which respects joins in both variables and satisfies $(ab)m = a(bm)$ for all $a, b \in Q, m \in M$. *Right Q -modules* are defined similarly and a *Q -bimodule* is required to also satisfy $(am)b = a(mb)$ for all $a, b \in Q, m \in M$.

We say that a left Q -module M is: *unital* if Q is unital and $em = m$ for every $m \in M$; *strong* if $1_Q m = 1_M$ for every $m \in M, m \neq 0$. The right adjoints of $a \cdot (\)$ and $(\) \cdot a$ on a Q -bimodule M are also denoted $(\) \leftarrow a, a \rightarrow (\)$. We also write $m \rightarrow n = \bigvee \{a \in Q \mid am \leq n\}$, $m \leftarrow n = \bigvee \{a \in Q \mid na \leq m\}$ for $m, n \in M$.

A *homomorphism of left Q -modules* $f : M \rightarrow N$ is a sup-homomorphism which satisfies $f(am) = af(m)$ for every $a \in Q, m \in M$. Homomorphisms of right Q -modules and Q -bimodules are defined in a similar way.

An important example is $\mathcal{Q}(S)$, the quantale of endomorphisms of a fixed sup-lattice S with composition as multiplication and suprema calculated pointwise. Its right- and left-sided elements are those of the form

$$\rho_x(y) = \begin{cases} x, & y \neq 0, \\ 0, & y = 0, \end{cases} \quad \lambda_x(y) = \begin{cases} 1, & y \not\leq x, \\ 0, & y \leq x, \end{cases}$$

hence $\mathcal{T}\mathcal{Q}(S) = \mathbf{2}$; $\mathcal{Q}(S)$ is von Neumann because $\rho_x^\perp = \lambda_x, \lambda_x^\perp = \rho_x$. Moreover $\mathcal{Q}(S)$ is *simple* [14, 12] and every element $\alpha \in \mathcal{Q}(S)$ can be expressed as below.

$$\alpha = \bigwedge_{x \in S} (\rho_{\alpha(x)} \vee \lambda_x) = \bigwedge_{x \in S} (\rho_x \vee \lambda_{\alpha^\perp(x)})$$

An element d of a quantale Q is called: *cyclic* if $ab \leq d \Leftrightarrow ba \leq d$ for all $a, b \in Q$; *dualizing* if $d \leftarrow (a \rightarrow d) = (d \leftarrow a) \rightarrow d = a$ for every $a \in Q$. Q is called *Girard* if it has a cyclic dualizing element. In that case we write $a^\perp = a \rightarrow d = d \leftarrow a$.

The *spectrum* of a C*-algebra A , denoted $\text{Max } A$, is the sup-lattice of all linear subspaces of A which are closed with respect to the norm topology. It is a quantale with respect to the multiplication $ab = \text{cl}\{AB \mid A \in a, B \in b\}$. The right- and left-sided elements of $\text{Max } A$ are, respectively, the closed right and left ideals of A .

Given a von Neumann algebra $M \subseteq \mathcal{B}(H)$, one may consider either the *weak spectrum* $\text{Max}_w M$ or the *ultraweak spectrum* $\text{Max}_{\sigma w} M$; the former consists of all linear subspaces of M which are closed with respect to the weak (operator) topology, the latter of those which are closed with respect to the somewhat finer ultraweak topology. The weak spectrum of a von Neumann algebra is a von Neumann quantale [13]; it follows that the same is true for ultraweak spectra, which are better suited to our purposes. [It is well-known that an ideal is ultraweakly closed if and only if it is weakly closed.]

We recall that a functional $\mathcal{B}(H) \rightarrow \mathbb{C}$ is ultraweakly continuous if and only if it has the form $\sum_{i=1}^{\infty} x_i(\phi_i, (\cdot) \cdot \psi_i)$ for some orthonormal families $\phi_i, \psi_i \in H$ and coefficients $x_i \in \mathbb{C}$ such that $\sum |x_i| < \infty$. Moreover, a subspace of $\mathcal{B}(H)$ is ultraweakly closed if and only if it is the intersection of the kernels of some set of ultraweakly continuous functionals.

An element $C \in \mathcal{B}(H)$ is said to be *trace-class* if

$$\|C\|_1 = \sum_{i=1}^{\infty} (\phi_i, \sqrt{C^*C} \phi_i) < \infty$$

for some orthonormal basis ϕ_i of H . The set of all trace-class elements is an ideal in $\mathcal{B}(H)$ and is denoted $\mathcal{C}_1(H)$. The number $\|C\|_1$ does not depend on the chosen basis and defines a norm on $\mathcal{C}_1(H)$. The $\|\cdot\|_1$ -closed subspaces form a spectrum $\text{Max}_1 \mathcal{C}_1(H)$.

The more general *Schatten class* $\mathcal{C}_p(H)$ for $p \geq 1$ is defined as a set of all elements $C \in \mathcal{B}(H)$ such that

$$\|C\|_p = \left(\sum_{i=1}^{\infty} (\phi_i, \sqrt{C^*C} \phi_i)^p \right)^{1/p} < \infty;$$

subspaces closed in the $\|\cdot\|_p$ -norm form a spectrum, $\text{Max}_p \mathcal{C}_p(H)$.

Given a family of Hilbert spaces H_i , we can think of the algebra $\prod \mathcal{B}(H_i)$ as that subalgebra of $\mathcal{B}(\bigoplus H_i)$ consisting of those operators for which H_i are invariant subspaces; similarly, $\bigoplus \mathcal{C}_1(H_i) = \mathcal{C}_1(\bigoplus H_i) \cap \prod \mathcal{B}(H_i)$. The induced topologies on $\prod \mathcal{B}(H_i) \subseteq \mathcal{B}(\bigoplus H_i)$, $\bigoplus \mathcal{C}_1(H_i) \subseteq \mathcal{C}_1(\bigoplus H_i)$ allow us to define spectra $\text{Max}_{\sigma w} \prod \mathcal{B}(H_i)$, $\text{Max}_1 \bigoplus \mathcal{C}_1(H_i)$.

The algebra of $n \times n$ complex matrices, which is isomorphic to $\mathcal{B}(\mathbb{C}^n)$, is denoted $M_n\mathbb{C}$.

3 Girard couples

1 Definition. A *couple (of quantales)* consists of two quantales C, Q together with a *coupling map* $\phi : C \rightarrow Q$ such that C is also a Q -bimodule, ϕ is a Q -bimodule homomorphism, and

$$\phi(c_1)c_2 = c_1\phi(c_2) = c_1c_2 \quad (*)$$

holds for all $c_1, c_2 \in C$.

Assume that $C \xrightarrow{\phi} Q$ is a couple. An element $d \in C$ is said to be *cyclic* if $ac \leq d \Leftrightarrow ca \leq d$ for all $a \in Q, c \in C$. The element d is said to be *dualizing* if $d \leftarrow (a \rightarrow d) = (d \leftarrow a) \rightarrow d = a$ for all $a \in Q$ and $d \leftarrow (c \rightarrow d) = (d \leftarrow c) \rightarrow d = c$ for all $c \in C$. In the case where d is both cyclic and dualizing we write $a \perp c \Leftrightarrow ac \leq d \Leftrightarrow ca \leq d$ for $a \in Q, c \in C$ and $a^\perp = a \rightarrow d = d \leftarrow a = \bigvee \{c \in C \mid a \perp c\}, c^\perp = c \rightarrow d = d \leftarrow c = \bigvee \{a \in A \mid a \perp c\}$.

A couple $C \xrightarrow{\phi} Q$ is said to be: *strong* if ϕ is strong; *unital* if Q is a unital quantale and C is a unital Q -bimodule; *Girard* if it has a cyclic dualizing element.

2 Example. (1) $Q \xrightarrow{id} Q$ is clearly a strong couple for any quantale Q . It is unital, or Girard, if and only if Q is unital, or Girard, respectively.

(2) Given an arbitrary unital quantale Q , we can construct a Girard couple $Q^{\text{op}} \xrightarrow{0} Q$ as follows: Q^{op} is equipped with the zero multiplication and the Q -bimodule structure given by $ac = (a \rightarrow c')', ca = (c' \leftarrow a)'$; 0 is the constantly zero map; the cyclic dualising element is $e' \in Q^{\text{op}}$. [Recall that we use $'$ to distinguish elements of Q and Q^{op} .] Indeed, $a^\perp = a \rightarrow e' = \bigvee \{c \mid ca \leq e'\} = \bigvee \{c \mid c' \leftarrow a \geq e\} = \bigvee \{c \mid a \leq c'\} = a'$. This couple is clearly not strong unless $Q = \{0\}$.

(3) If $C_j \xrightarrow{\phi_j} Q_j$ are couples, then so is $\prod_j C_j \xrightarrow{(\phi_j)_j} \prod_j Q_j$. Moreover, it is strong, unital, or Girard, if and only if each component is so.

(4) Let R be a (unital) ring, I a two-sided ideal, and $\text{Sub } R, \text{Sub } I$ the quantales of their additive subgroups. Then $\text{Sub } I \subseteq \text{Sub } R$ is a (unital) couple.

3 Proposition. Let $C \xrightarrow{\phi} Q$ be a couple. Then

(1) $a(c_1c_2) = (ac_1)c_2$, $(c_1c_2)a = c_1(c_2a)$ and $(c_1a)c_2 = c_1(ac_2)$ for all $a \in Q, c_1, c_2 \in C$.

(2) $\phi : C \rightarrow Q$ is a quantale homomorphism.

Proof. (1) Using $(*)$ twice, and the fact that C is a Q -bimodule, we obtain $a(c_1c_2) = a(c_1\phi(c_2)) = (ac_1)\phi(c_2) = (ac_1)c_2$. The proofs of the other two equations are similar.

(2) Using $(*)$ and the fact that ϕ is left Q -module homomorphism, we have $\phi(c_1c_2) = \phi(\phi(c_1)c_2) = \phi(c_1)\phi(c_2)$. \square

4 Remark. Let \mathcal{J} denote the two-element chain, now regarded not as an object of \mathcal{Sup} but as a monoidal category in its own right (with $\otimes = \wedge$), and let $! : 0 \rightarrow 1$ denote the unique non-identity morphism of \mathcal{J} . Then monoidal functors $F : \mathcal{J} \rightarrow \mathcal{Sup}$ are in bijective correspondence with unital couples of quantales $C \xrightarrow{\phi} Q$. [By way of comparison, recall that a unital quantale is equivalent to a monoid in \mathcal{Sup} which, in turn, is equivalent to a monoidal functor $\mathcal{T} \rightarrow \mathcal{Sup}$ where \mathcal{T} is the terminal category.]

The correspondence is given by $C = F_0$, $Q = F_1$, $\phi = F_!$. The *multiplication natural transformation* of F encompasses all four binary operations of the couple (e.g., its $(1, 0)$ -component, $F_1 \otimes F_0 \rightarrow F_{1 \otimes 0} = F_0$, corresponds to the left action of Q on C); its naturality is equivalent to the restrictions placed on ϕ ; the pentagon which it is required to satisfy summarises all the associativity conditions which a couple satisfies, including those of Proposition 3(1). Similarly, the *unit arrow* of F , which must have the form $\mathbf{2} \rightarrow F_1$, picks out an element of Q ; the triangles which it is required to satisfy assert not only that this be a unit for Q but also that it act as a unit on C .

A very abstract approach to dualising elements, which can be applied to a much larger class of monoidal functors, is discussed in a parallel paper [4]. Much of what follows for couples of quantales remains true in the more general setting.

5 Proposition. Let $C \xrightarrow{\phi} Q$ be a Girard couple. Then ϕ is self-adjoint, i.e. $\phi^* = \phi$.

Proof. The assertion follows from $c_1 \leq \phi^\perp(c_2^\perp) \Leftrightarrow \phi(c_1) \leq c_2^\perp \Leftrightarrow \phi(c_1)c_2 = c_1\phi(c_2) \leq d \Leftrightarrow c_1 \leq \phi(c_2)^\perp$. \square

6 Remark. Given a Girard couple $C \xrightarrow{\phi} Q$, one can define $a \sqcup b = (b^\perp a^\perp)^\perp$ for $a, b \in C \cup Q$. The four resultant operations all correspond to the *multiplicative join* alias *par* of linear logic. Collectively, they give $Q^{\text{op}} \xrightarrow{\phi^*} C^{\text{op}}$

the structure of a Girard couple, with neutral element d' and cyclic dualising element e' ; by the previous proposition, this is isomorphic, as a Girard couple, to $C \xrightarrow{\phi} Q$.

7 Proposition. *Let $C \xrightarrow{\phi} Q$ be a strong couple of semiunital quantales. Then ϕ is an isomorphism on right- and left-sided elements.*

Proof. We prove the right-sided case. Let $r \in \mathcal{R}(Q), s \in \mathcal{R}(C)$. Then $\phi(r1_C) = r\phi(1_C) = r1_Q = r$ and $\phi(s)1_C = s1_C = s$, hence $\phi|_{\mathcal{R}(C)}$ and $(\) \cdot 1_C$ are mutually inverse sup-homomorphisms. \square

8 Proposition. *A Girard couple $C \xrightarrow{\phi} Q$ is unital; if it is also strong, then both C and Q are von Neumann quantales.*

Proof. Let $d \in C$ be a cyclic dualizing element. All the equalities of [16, Proposition 6.1.2] can be easily adapted for Girard couples; in particular, $e = d^\perp$ is a unit for Q .

Now assume that $r \leq d$ for some $r \in \mathcal{R}(C)$; then $r1_C = r1_Q \leq d$, and hence $r \leq 1_Q^\perp = 0_C$. That is, the only right- or left-sided element below d is 0. It follows that, for all pairs $r \in \mathcal{R}(C), l \in \mathcal{L}(Q)$, and for all pairs $r \in \mathcal{R}(Q), l \in \mathcal{L}(C)$, $alr \leq d \Leftrightarrow lr = 0$. Thus $lr = l\phi(r) = \phi(l)r$ for $r \in \mathcal{R}(C), l \in \mathcal{L}(C)$ and Proposition 7 entail that C is von Neumann. The previous proposition also entails $lr = 0_Q \Leftrightarrow \phi^\perp(l)\phi^\perp(r) = \phi^\perp(0_Q) = 0_C$ for $r \in \mathcal{R}(Q), l \in \mathcal{L}(Q)$; hence Q is also von Neumann. \square

9 Corollary. *Every Girard quantale is von Neumann and the Girard duality extends the von Neumann duality.*

10 Theorem. *Let S be a sup-lattice. Then the assignment*

$$(x \otimes y')(u \otimes v') = \begin{cases} 0 & \text{if } u \leq y, \\ x \otimes v' & \text{otherwise} \end{cases}$$

defines a quantale structure on $S \otimes S^{\text{op}}$ which will be denoted $\mathcal{C}(S)$.

The assignment

$$\phi(x \otimes y') = \rho_x \lambda_y$$

defines a strong Girard couple $\mathcal{C}(S) \xrightarrow{\phi} \mathcal{Q}(S)$ with a cyclic dualizing element

$$d = \bigvee_{x \in S} (x \otimes x').$$

Proof. The given binary operation is clearly associative and distributive on generators of $S \otimes S^{\text{op}}$. For example,

$$\begin{aligned} (x \otimes \bigvee y'_i)(u \otimes v') &= (x \otimes (\bigwedge y_i'))(u \otimes v') = \begin{cases} 0 & \text{if } \forall i \ u \leq y_i \\ x \otimes v' & \text{otherwise} \end{cases} \\ &= \bigvee (x \otimes y'_i)(u \otimes v'). \end{aligned}$$

Thus, by the definition of \otimes , it extends to all elements of $S \otimes S^{\text{op}}$.

ϕ too is clearly a well-defined sup-homomorphism since it is “bilinear” on generators. It is strong because $\phi(1_{\mathcal{C}(S)}) = \phi(1 \otimes 0') = \rho_1 \lambda_0 = 1_{\mathcal{Q}(S)}$.

S is a left $\mathcal{Q}(S)$ -module with action $\alpha x = \alpha(x)$ and S^{op} is a right $\mathcal{Q}(S)$ -module with action $y' \alpha = \alpha^{-1}(y)'$. Therefore $S \otimes S^{\text{op}}$ carries $\mathcal{Q}(S)$ -bimodule structure. The axiom $(*)$ is obtained as follows

$$\begin{aligned} \phi(x \otimes y')(u \otimes v') &= \rho_x \lambda_y(u) \otimes v' \\ &= \begin{cases} 0 & \text{if } u \leq y, \\ x \otimes v' & \text{otherwise} \end{cases} \\ &= (x \otimes y')(u \otimes v') \end{aligned}$$

and symmetrically for $(x \otimes y')\phi(u \otimes v')$. Since the operations of $\mathcal{Q}(S)$ are given pointwise, the remaining axioms of a couple are evident.

The duality $\mathcal{C}(S) \cong \mathcal{Q}(S)^{\text{op}}$ was proven in [6] and is given by

$$(\lambda_x \vee \rho_y)^\perp = x \otimes y'.$$

Namely, for $\alpha \in \mathcal{Q}(S), c \in \mathcal{C}(S)$ we have $\alpha \perp c$ when $x \otimes y' \leq c$ implies that $\alpha \leq \lambda_x \vee \rho_y$. We will show that d is a cyclic dualizing element of the couple $\mathcal{C}(S) \xrightarrow{\phi} \mathcal{Q}(S)$. We can see that

$$(\lambda_x \vee \rho_y)(u \otimes v') \leq d \Leftrightarrow u \leq x \text{ and } y \leq v \Leftrightarrow (u \otimes v')(\lambda_x \vee \rho_y) \leq d$$

whenever $(\lambda_x \vee \rho_y) \neq 1, (u \otimes v') \neq 0$ and

$$1(u \otimes v') \leq d \Leftrightarrow u \otimes v' = 0 \Leftrightarrow (u \otimes v')1 \leq d.$$

Hence

$$(x \otimes y') \rightarrow d = d \leftarrow (x \otimes y') = \lambda_x \vee \rho_y$$

and

$$(\lambda_x \vee \rho_y) \rightarrow d = x \otimes y' = d \leftarrow (\lambda_x \vee \rho_y).$$

Consequently,

$$\alpha \rightarrow d = \bigvee_{x \in S} (\alpha^\perp(x) \otimes x') = \left(\bigwedge_{x \in S} (\lambda_{\alpha^\perp(x)} \vee \rho_x) \right)^\perp = \alpha^\perp$$

and similarly

$$d \leftarrow \alpha = \bigvee_{x \in S} (\alpha(x) \otimes x') = \alpha^\perp$$

for every $\alpha \in \mathcal{Q}(S)$. The inverse duality $^\perp : \mathcal{C}(S) \rightarrow \mathcal{Q}(S)$ follows directly from a general property of adjoints $(\bigvee c_i) \rightarrow d = \bigwedge (c_i \rightarrow d)$. Thus d is a cyclic dualizing element of $\mathcal{C}(S) \xrightarrow{\phi} \mathcal{Q}(S)$. \square

We remark that the morphism $\phi : \mathcal{C}(S) \rightarrow \mathcal{Q}(S)$ is an instance of a *mix map* [3]. G. N. Raney [15] proved that this ϕ is an isomorphism if and only if S satisfies *complete distributivity*:

$$\bigwedge_{j \in J} \bigvee_{k \in K} \alpha_{jk} = \bigvee_{f \in K^J} \bigwedge_{j \in J} \alpha_{jf(j)}.$$

We obtain the following statement which has already been mentioned in [8].

11 Corollary. *$\mathcal{Q}(S)$ is a Girard quantale if and only if S is completely distributive.*

12 Theorem. *Let H be a Hilbert space. Then the assignment $\phi(c) = \text{cl}_{\sigma_w}(c)$ (i.e. the ultraweak closure) defines a Girard couple $\text{Max}_1 \mathcal{C}_1(H) \xrightarrow{\phi} \text{Max}_{\sigma_w} \mathcal{B}(H)$ with a cyclic dualizing element*

$$d = \{C \in \mathcal{C}_1(H) \mid \text{tr } C = 0\}.$$

Proof. The basic idea is that $(A, C) \mapsto \text{tr}(AC) = \text{tr}(CA)$ is a bilinear form on $\mathcal{B}(H) \times \mathcal{C}_1(H) \rightarrow \mathbb{C}$ which is continuous in each variable with respect to the appropriate topology.

It is known [7] that the ultraweakly continuous functionals on $\mathcal{B}(H)$ are of the form $\text{tr}(C \cdot ())$ for some $C \in \mathcal{C}_1(H)$, and conversely, the $\|\cdot\|_1$ -norm continuous functionals on $\mathcal{C}_1(H)$ are of the form $\text{tr}(A \cdot ())$ for some $A \in \mathcal{B}(H)$. In the spectra, operators correspond to atoms and functionals to coatoms. More precisely, we work with one-dimensional subspaces and kernels of functionals. Every closed subspace (in the topology considered) can then be obtained as a join of atoms or meet of coatoms and it is known

that the families of atoms and coatoms separate each other. From this fact it follows that the assignment

$$a \perp c \Leftrightarrow (\forall A \in a, C \in c \text{ tr}(AC) = 0)$$

admits a duality between $\text{Max}_{\sigma w} \mathcal{B}(H)$ and $\text{Max}_1 \mathcal{C}_1(H)$. Moreover, the trace is symmetric on operators and thus also on subspaces, *i.e.* d is cyclic and from the duality it follows that d is dualizing. We obtain

$$a^\perp = \bigwedge \{ \ker \text{tr}(A \cdot ()) \mid A \in a \}.$$

$\text{Max}_1 \mathcal{C}_1(H)$ is a $\text{Max}_{\sigma w} \mathcal{B}(H)$ -bimodule since $\mathcal{C}_1(H)$ is a two-sided ideal in $\mathcal{B}(H)$ and both multiplications $\mathcal{B}(H) \times \mathcal{C}_1(H) \rightarrow \mathcal{C}_1(H)$, $\mathcal{C}_1(H) \times \mathcal{B}(H) \rightarrow \mathcal{C}_1(H)$ are continuous. The ultraweak topology is weaker than the $\| \cdot \|_1$ -norm topology, thus it defines a closure on $\text{Max}_1 \mathcal{C}_1(H)$. From continuity it follows again that $\text{cl}_{\sigma w}(ac) = a \text{cl}_{\sigma w}(c)$ and hence ϕ is a bimodule homomorphism. It is strong because $\text{cl}_{\sigma w} \mathcal{C}_1(H) = \mathcal{B}(H)$. \square

13 Corollary. *The spectrum $\text{Max } M_n \mathbb{C}$ is a Girard quantale.*

Proof. On a finite-dimensional Hilbert space all operators are trace-class and the norm, $\| \cdot \|_1$ -norm, and ultraweak topologies coincide, hence ϕ is an isomorphism. \square

14 Proposition. *Let H_i be a family of Hilbert spaces. Then $\text{Max}_1 \bigoplus \mathcal{C}_1(H_i) \xrightarrow{\phi} \text{Max}_{\sigma w} \prod \mathcal{B}(H_i)$ is a strong Girard couple.*

In particular, the spectrum $\text{Max } A$ of a finite-dimensional C^ -algebra A is a Girard quantale.*

Proof. Since $\prod \mathcal{B}(H_i) \subseteq \mathcal{B}(\bigoplus H_i)$, $\bigoplus \mathcal{C}_1(H_i) \subseteq \mathcal{C}_1(\bigoplus H_i)$ are closed subalgebras, we can correctly restrict $\phi : \text{Max}_1 \mathcal{C}_1(\bigoplus H_i) \rightarrow \text{Max}_{\sigma w} \mathcal{B}(\bigoplus H_i)$ to $\phi|_{\text{Max}_1 \bigoplus \mathcal{C}_1(H_i)} : \text{Max}_1 \bigoplus \mathcal{C}_1(H_i) \rightarrow \text{Max}_{\sigma w} \prod \mathcal{B}(H_i)$. Then all calculations are made with respect to the invariant subspaces H_i and $\{C \in \bigoplus \mathcal{C}_1(H_i) \mid \text{tr}(C) = 0\}$ provides a duality between elements of $\bigoplus \mathcal{C}_1(H_i)$ and ultraweakly continuous functionals restricted to $\prod \mathcal{B}(H_i)$. It is true that $\text{cl}_{\sigma w}(\bigoplus \mathcal{C}_1(H_i)) = \prod \mathcal{B}(H_i)$. The rest follows from Theorem 12.

Finite-dimensional C^* -algebras are of the form $\prod_{i=1}^k M_{n_i} \mathbb{C} = \bigoplus_{i=1}^k M_{n_i} \mathbb{C}$, hence the assertion. \square

15 Theorem. *Let $C \xrightarrow{\phi} Q$ be a Girard couple. Then ϕ factors through a Girard quantale G , *i.e.* there are quantale homomorphisms $\gamma : C \rightarrow G$ and*

$\alpha : G \rightarrow Q$ such that $\phi = \alpha\gamma$. Moreover, $C \xrightarrow{\gamma} G$ is a couple and the G -module actions are given by restricting scalars along α :

$$gc = \alpha(g)c, \quad cg = c\alpha(g)$$

for all $g \in G, c \in C$.

If ϕ is strong then γ, α can be chosen to be strong. Consequently, $\mathcal{R}(C) \cong \mathcal{R}(G) \cong \mathcal{R}(Q), \mathcal{L}(C) \cong \mathcal{L}(G) \cong \mathcal{L}(Q)$.

Proof. We follow the idea of [16, Theorem 6.1.3]. Let $G = \{(a, c) \in A \times C \mid \phi(c) \leq a\}$ with joins given componentwise and multiplication $(a_1, c_1)(a_2, c_2) = (a_1a_2, a_1c_2 \vee c_1a_2)$. From definition of a couple we easily check that $A \times C$ is a quantale. G is clearly closed under joins and from $\phi(a_1c_2 \vee c_1a_2) = a_1\phi(c_2) \vee \phi(c_1)a_2 \leq a_1a_2$ it follows that it is a strong subquantale of $A \times C$.

Put $\gamma(c) = (\phi(c), c), \alpha(a, c) = a$. The projection α is evidently a strong quantale homomorphism. Actions $c_1(a, c_2) = c_1\gamma(a, C_2) = c_1a, (a, c_1)c_2 = \gamma(a, c_1)c_2 = ac_2$ define a G -bimodule structure on C . Then $\gamma(c_1(a, c_2)) = \gamma(c_1a) = (\phi(c_1a), c_1a) = (\phi(c_1)a, c_1a) = \gamma(c_1)(a, c_2)$ because $\phi(c_1)c_2 = c_1c_2 \leq ac_2$. Similarly we can check the other side and hence γ is a G -bimodule homomorphism. From the properties of ϕ it follows that $C \xrightarrow{\gamma} G$ is also a couple. The largest element of G is $(1_Q, 1_C)$, thus γ is strong whenever ϕ is so.

Finally, $(1, d)$ is a cyclic element of G and $(e, 0)$ is a unit. Indeed, we have $(a_1, c_1)(a_2, c_2) \leq (1, d) \Leftrightarrow a_1c_2 \vee c_1a_2 \leq d \Leftrightarrow a_1 \perp c_2$ and $c_1 \perp a_2$ which yields the awaited duality $(a, c)^\perp = (c^\perp, a^\perp)$.

The rest follows from Proposition 7. \square

16 Corollary. For every sup-lattice S there exists a Girard quantale $\mathcal{G}(S)$ with $\mathcal{RG}(S) \cong S, \mathcal{LG}(S) \cong S^{\text{op}}$.

17 Remark. (1) Rosenthal's Girard quantale $Q \times Q^{\text{op}}$ ([16, Theorem 6.1.3]) arises as our G for the zero Girard couple $Q^{\text{op}} \xrightarrow{0} Q$ of Example 2(2).

(2) The multiplication in $\mathcal{G}(S)$ can be interpreted as a convolution product: let $a = (a_0, a_1), b = (b_0, b_1) \in \mathcal{G}(S)$, then

$$(ab)_i = \bigvee_{j \wedge k \leq i} a_j b_k$$

for both $i \in \{0, 1\}$.

18 Example. It is possible to meld all the spectra $(\text{Max}_p \mathcal{C}_p(H))_{p \in [1, \infty]}$ (where $\text{Max}_\infty \mathcal{C}_\infty H := \text{Max}_{\sigma_w} \mathcal{B}(H)$) into a single monoidal functor $F : [0, 1] \rightarrow \text{Sup}$, thus extending the framework of Remark 4.

Here $[0, 1]$ is regarded as a thin monoidal category with the *Lukasiewicz multiplication*

$$i \&_L j = \max\{0, i + j - 1\}$$

so that if $p, q, r \in [1, \infty]$ satisfy

$$\frac{1}{r} = \min\left\{1, \frac{1}{p} + \frac{1}{q}\right\} \quad (\dagger)$$

and if f denotes the bijection $[1, \infty] \rightarrow [0, 1]$ given by $f(p) = 1 - \frac{1}{p}$, $f^{-1}(i) = \frac{1}{1-i}$, then $f(p) \&_L f(q) = f(r)$.

The functor F is given by $F_i = \text{Max}_{f^{-1}(i)} \mathcal{C}_{f^{-1}(i)}(H)$ and $F(i \rightarrow j) = \text{cl}_{f^{-1}(j)}$ (*i.e.* the $\|\cdot\|_{f^{-1}(j)}$ -norm closure), and its multiplication natural transformation by the well-known fact that $A \in \mathcal{C}_p(H), B \in \mathcal{C}_q(H)$ implies $AB \in \mathcal{C}_r(H)$ where r is determined by (\dagger) .

Moreover it is possible to construct a single Girard quantale G from this data, analogous to that constructed in Theorem 15:

$$G = \left\{ (a_i) \in \prod_{i \in [0,1]} \text{Max}_{f^{-1}(i)} \mathcal{C}_{f^{-1}(i)}(H) \mid \text{cl}_{f^{-1}(j)} a_i \subseteq a_j \text{ for } i \leq j \right\},$$

$$(ab)_i = \bigvee_{j \&_L k \leq i} \text{cl}_{f^{-1}(i)}(a_j b_k).$$

19 Open problems. We have shown that the spectra of the operator algebras $\mathcal{B}(H)$ (and their products) together with the spectra of their preduals from Girard couples. It is natural to ask whether our results can be generalized for all W^* -algebras, including the ideas of the previous example. If so, it would be useful to describe essential concepts of W^* -algebras, *e.g.* normal morphisms, by means of the discussed monoidal functors.

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